

On the complexity of fill minimization for nonsymmetric matrices

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Abstract

The complexity of choosing a permutation matrix P to minimize the fill in the LU factors of PAP^T is known; this is not the case if the row and column permutations are allowed to differ. We show that the following constrained version of the problem is NP-complete: given a matrix A and a constant k , does there exist a permutation matrix Q such that the LU factorization of AQ^T generates no more than k fill elements.

Introduction

The fill introduced when computing the LU factorization of a sparse matrix A can often be reduced by first permuting the rows and/or columns of A . In other words, it may be possible to find permutation matrices P and Q such that the LU factors of PAQ^T are sparser than the LU factors of A . We ask how hard it is to compute the optimal fill-minimizing permutation matrices in the situation where A may be nonsymmetric.

If A is structurally symmetric, or if there is some other reason to keep the diagonal elements of A on the diagonal of PAQ^T , it may make sense to require that $P = Q$. In this situation, choosing P to minimize the fill is known to be an NP-complete problem both for arbitrary A [1, 3] and for structurally symmetric A [4]. However, showing NP-completeness in the situation where P and Q are allowed to differ has proven more challenging.

In this talk we will discuss the general nonsymmetric fill-minimization problem and give some insight into why it has proven so challenging, then describe a proof that the following one-sided, nonsymmetric version of the minimum fill problem is NP-complete:

$$\text{MINFILLCOL} = \{ \langle A, k \rangle \mid \exists Q \text{ such that LU factorization on } AQ^T \\ \text{without pivoting introduces no more than } k \text{ fill} \}$$

Graph formulation

As described in [2] and elsewhere, the minimum fill problem on matrices can be interpreted in terms of a graph problem. If $P = Q$, the minimum fill problem is equivalent to choosing an ordering on the vertices of a graph G , where G has A as an adjacency matrix. If P and Q can differ, the problem corresponds to choosing an ordering on pairs of vertices in the directed graph G , where again G has A as an adjacency matrix.

Focussing on the MINFILLCOL problem, if you are given the directed graph $G = (V, E)$ whose adjacency matrix is A , then fixing the row ordering corresponds to assigning a total ordering on

the vertices v_1, v_2, \dots, v_n . Applying a column permutation Q and determining the fill generated in the subsequent factorization can be done as follows:

1. let $E_1 = E, G_1 = (V, E_1)$
2. **for** $i = 1$ to n
3. choose an edge $e_i = (v_i, v_j) \in E_i$
4. $E'_i = E_i \cup \{(v_r, v_k) | (v_i, v_k) \in E_i \text{ and } (v_r, v_j) \in E_i\}$
5. $E_{i+1} = E'_i - \{(v_s, v_t) | s = i \text{ or } t = j\}$
6. **endfor**

The fill consists of the edges added over the n iterations of line 4.

The MINFILLCOL problem asks whether, given the ordering v_1, v_2, \dots, v_n , there exists a sequence of edges e_1, e_2, \dots, e_n such that no more than k fill is generated.

Results

Theorem 1. MINFILLCOL is NP-complete.

The reduction is from SAT-NOR, a version of SAT in which the boolean expression contains only nor operators. The proof describes how to construct a graph G' from an instance of SAT-NOR with n variables and m nor gates. If A is an adjacency matrix for G' , then $\langle A, 2n + 28m - 8 \rangle$ is a yes instance of MINFILLCOL if and only if the original SAT-NOR expression is satisfiable.

References

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