Algebraic Distance and Its Applications to Combinatorial Scientific Computing Problems

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Extended Abstract

This work addresses general questions related to any graph model: Given a graph G and two pairs of vertices (i, j) and (u, v), which pair of vertices is "connected" better and how can this "connectivity" be used in combinatorial optimization problems? Our goal is to design a process for measuring connectivity that is fast, can be parallelized, easily implemented and can be applied on local parts of the data when there is no need to estimate the connectivity over all data.

Since the notion of connectivity is of practical significance, many algorithms have been developed in order to model it. In a random-walk approach [7, 11], the average first-passage time/cost and average commute time were used. A similarity measure between nodes of a graph integrating indirect paths, based on the matrix-forest theorem, was proposed in [4]. Approximation of a betweenness centrality in [1] allows one to apply this computationally expensive method. A convergence of the compatible relaxation [2] was measured in AMG schemes [9] in order to detect strong connections between fine and coarse points. A similarity method based on probabilistic interpretation of a diffusion was introduced in [8].

Algebraic distance. Consider a Jacobi stationary iterative method for solving a linear system $AX = (D + L + U)X = B$, where D, L, and U correspond to the diagonal, lower triangular part and upper triangular part of A, respectively. Its $(k + 1)$th iteration is defined by the following scheme,

$$X^{(k+1)}_{JAC} = D^{-1}(B - (L + U)X^{(k)}_{JAC}) \ , \tag{1}$$

and the corresponding successive overrelaxation (SOR) by

$$X^{(k+1)}_{NEW\ JAC} = (1 - \omega)X^{(k)} + \omega X^{(k+1)}_{JAC} \ , \tag{2}$$

where ω is a convergence acceleration parameter. We will use the first t iterations of Jacobi ω-SOR in order to determine the weakly connected edges of the graph. Let A be the graph Laplacian, $B = 0^{n\times r}$, and random matrix $X^{(0)} \in_R (-\frac{1}{2}, \frac{1}{2})^{n\times r}$.

Definition. *Given a graph G and assuming X is an outcome of t iterations of (2), then the algebraic distance between nodes i and j is a function $\rho : V \times V \to (0, 1)$ defined by*

$$\rho_{ij} = \max_{l=1}^{r}\{|x_i^{\mathbf{l}} - x_j^{\mathbf{l}}|\} \ .$$

One can prove that if for every i, j, and l $|x_i^{\mathbf{l}} - x_j^{\mathbf{l}}| \neq 0$, then ρ is a *metric* on the graph. The introduced notion of *algebraic distance* is a generalization based on the principle of obtaining low-residual error components used in the Bootstrap AMG [3]. When a priori knowledge of the nature of this error is not available, slightly relaxed random vectors are used to approximate it.

Similarly, we will introduce an algebraic distance based on *Gauss-Seidel* relaxation and *non-symmetric* relaxations. In this talk, we will discuss the connectivity properties of ρ and its applications.

Multilevel algorithms for combinatorial problems. The need for an improved measure for the graph couplings can be explained by observing the graph depicted in Fig 1: one additional edge ij (connecting nodes i and j) is added to a regular mesh. For many connectivity-based problems (such

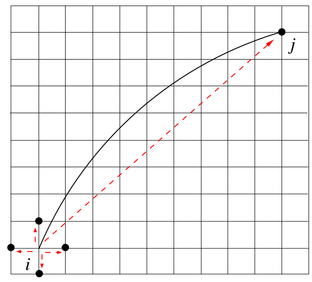

Figure 1: Mesh graph with an additional edge between nodes i and j. The black dots mark some of the nodes selected to serve as the seeds of the coarse aggregates.

as linear ordering and partitioning), while coarsening, nodes i and j should not belong to the same aggregate. However, if the weight of ij is slightly larger than other ik connections and if the black dots are some of the seeds of the coarse aggregates (chosen by some AMG-based coarsening), node i clearly will be attracted to node j. Similar behavior can be observed in any edge-matching multilevel scheme. Such a decision will create a local conflict; moreover, at the next coarse levels, this conflict may be reinforced by making similar local wrong decisions. To prevent this situation, we would like to have a measure that evaluates the coupling between i and j, not only according to the *direct* coupling between them but also taking into account the contribution of connections between the *neighborhoods* of i and j.

For this purpose, we will show how to design a ρ-based coarsening and will demonstrate the results of the comparison of classical AMG with ρ-based coarsening for the following problems:

Problem	Improvement up to
minimum linear arrangement	40%
minimum 2-sum	45%
minimum bandwidth	20%
minimum 2-partitioning	30%

The improvement was mostly on the graphs that are not obtained from the finite element instances. Comparison of the classical AMG scheme with other methods was done earlier in [10, 5] and is outside the scope of this work.

Maximum weighted matching problem. To demonstrate the influence of the ρ-based connectivity measure, we used two 1/2-approximation algorithms (classical greedy algorithm and its fast version from [6]) as baseline frameworks. In both algorithms, a basic step of choosing a next edge for matching was reinforced by taking into account the corresponding ρ-based measure in which preference was given to the less connected components. The new heuristics were tested on a set of 150 graphs. The improvement was confirmed for almost all graphs. For 75% of them the matching was increased by more than 10%. For 10% of the test graphs the increase was between 20% and 45%.

Maximum independent set problem. Similarly to the previous problem, we reinforced a classical greedy algorithm for the maximum independent set by algebraic distance, taking into account the corresponding ρ-based measure in which preference was given to the less-connected nodes. The observed improvement was up to 20% with almost no worsening on a test set of 150 real-life graphs.

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